

# Earthquake Prediction: Probabilistic Aspect

G. MOLCHAN<sup>a,b</sup> and V. KEILIS-BOROK<sup>a,c</sup>

<sup>a</sup>*International Institute of Earthquake Prediction Theory and Mathematical Geophysics, Russian Academy of Sciences, Moscow, Russia. E-mail: molchan@mitp.ru(GM)*

<sup>b</sup>*The Abdus Salam International Centre for Theoretical Physics, SAND Group, Trieste, Italy*

<sup>c</sup>*Institute of Geophysics and Planetary Physics and Department of Earth and Space Sciences, University of California, Los Angeles, USA E-mail: vkb@ess.ucla.edu*

Version: 31 Jan. 2008

## S U M M A R Y

A theoretical analysis of the earthquake prediction problem in space-time is presented. We find an explicit structure of the optimal strategy and its relation to the generalized error diagram. This study is a generalization of the theoretical results for time prediction. The possibility and simplicity of this extension is due to the choice of the class of goal functions. We also discuss issues in forecasting versus prediction, scaling laws versus predictability, and measure of prediction efficiency at the research stage.

**Key words:** prediction, forecasting, error diagram, prediction efficiency.

# 1 Introduction

The sequence of papers [Molchan 1991, 1997, 2002] was an attempt at a probabilistic interpretation of what had been done in empirical earthquake prediction during the 1980-1990s. These papers deal with the problem of predicting the time of a large event in a fixed region.

The prediction involved the following concepts: *the information flow*  $I(t)$  used for prediction; *a prediction strategy*  $\pi$  consisting of a sequence of decisions  $\pi(t)$  that are relevant to consecutive time intervals  $(t, t + \Delta)$ ; *a decision*, which is based on the information  $I(t)$ , and which is to choose an alarm level for a time  $\Delta$  (the zero level means an absence of alarm); *losses*, which result from  $\pi(t)$  and depend on whether the decision is suitable for the actual seismic situation in  $\Delta$ ; *the goal of prediction*, which is to minimize a loss functional for the monitoring period  $T \gg 1$ .

In the general case the optimal strategy is found as the solution of a Bellman-type equation. However, there is one important case (at least, at the research stage of prediction) for which the optimal strategy is described explicitly, viz., the case where the goal function can be described in terms of known prediction characteristics: the rate of alarm time,  $\tau$ , and the rate of failures-to-predict,  $n$ . The optimal strategy is then described with the help of (a) conditional intensity of target earthquakes given  $I(t)$  and (b) the  $n$  &  $\tau$  (error) diagram,  $\Gamma$  (Fig. 1). The latter is defined as the low bound of the set of the prediction characteristics  $(n, \tau) \in [0, 1]^2$  that are relevant to all possible strategies  $\pi$  based on  $I = \{I(t)\}$ .

If the flow  $I$  is trivial, i.e., supplies no information for prediction, then  $\Gamma$  consists of the diagonal  $D$  of the square  $[0, 1]^2$ :  $n + \tau = 1$ . The curve  $\Gamma$  is a decreasing convex function. The greater the amount of information available, the larger is the distance between curves  $\Gamma$  and  $D$ . More precisely, the condition  $I_1(t) \subseteq I_2(t)$  implies  $\Gamma_1 \geq \Gamma_2$ . In the ideal case,  $\Gamma$  degenerates to the point  $n = \tau = 0$ .

In actual practice the target earthquakes are large, hence rare, events. This causes difficulties for statistical validation of a prediction algorithm in a small region. That difficulty is being overcome by parallel application of an algorithm in different regions (e. g. algorithm *M8* [Kossobokov and Shebalin, 2002] and RTP algorithm [Keilis-Borok et al., 2004]). Prediction results are, as before, presented using the error diagram, where  $\tau$  is replaced with the rate of space-time alarms  $\tilde{\tau}$ . The properties of the modified diagram have not been studied yet. Moreover, the generalization of  $\tau$  itself is not unique.

For example,  $\tilde{\tau}$  can be represented by the area of the alarm space  $A$  or by the expected number of target events within  $A$ , i.e.,  $\lambda(A)$ . Thus the case of space-time prediction needs analysis, and such an analysis is presented below (see Section 3).

Next, we also discuss two more issues: the relation between prediction and forecasting (sect. 4), and the relation between predictability and self-similarity (sect. 5). These issues seem to be urgent, considering that forecasting is dominant in prediction research today, and the scaling laws indicating self-similarity are frequently regarded as an obstacle in the way of predictability.

## 2 Time Prediction

Let us remind some facts concerning the simplest situation (see below) in predicting the time of a target event in a fixed region [Molchan, 2002].

The sequence of target events in the region will be considered as a random stationary point process  $dN(t)$ , where  $N(t)$  is the number of events in the interval  $(0, t)$  and  $P(\Delta N(t) \geq 2) = o(\Delta t)$ . The prediction of  $dN(t)$  is based on the information flow  $I(t)$ , such that the  $\{dN(t), I(t)\}$  form a stationary ergodic process;  $I(t)$  may be thought of as a catalog of earthquakes in a moving time interval  $(t - t_0, t - t_1)$  with  $t_0 > t_1 \geq 0$  fixed. A prediction strategy  $\pi = \{\pi(t)\}$  consists of a sequence of decisions  $\pi(t)$ :  $\pi = 1$  means an alarm during  $(t, t + \Delta)$ , while  $\pi = 0$  means an absence of alarm. The occurrence of a target earthquake during an alarm is termed a success. Each decision is based on  $I(t)$ . The strategies are stationary and related in a stationary manner to the process  $\{dN(t), I(t)\}$ .

The following prediction results are to be recorded during time  $T = S\Delta$ :

$$\tau_T = S^{-1} \sum_{k=1}^S \mathbf{1}_{\{\pi(t_k)=1\}}, \quad t_k = k \cdot \Delta \quad (1)$$

and

$$n_T = S^{-1} \sum_{k=1}^S \mathbf{1}_{\{\pi(t_k)=0\}} \mathbf{1}_{\{dN(t_k)=1\}} [S/N(T)]. \quad (2)$$

where the logical function  $\mathbf{1}_A$  equals 1 if  $A$  is true and 0 otherwise. These statistics determine the empirical rates of alarm time and failures-to-predict.

It follows from the above assumptions that  $\tau_T$  and  $n_T$  have deterministic limits  $\tau$  and  $n$ , respectively, as  $T \rightarrow \infty$ . They characterize the prediction capability of a strategy  $\pi$  based on the information  $I = \{I(t)\}$ . On the other hand, the  $n$  &  $\tau$  diagram mentioned in Introduction characterizes the prediction capability of  $I = \{I(t)\}$ .

Minimization of a goal function of type  $\varphi(n, \tau)$ , symbolically

$$\varphi(n, \tau) \Rightarrow \min_{\pi}, \quad (3)$$

is called here the *simplest prediction problem*. The choice of  $\varphi$  is governed by the particular applications of prediction considered. There are only two general limitations:  $\varphi$  should increase with increasing  $n$  and  $\tau$  and the level sets  $\{n, \tau : \varphi \leq c\}$  should be convex.

Typical examples of  $\varphi$  that are used at the research stage are  $\max(n, \tau)$  and  $n + \tau$ . The strategy that optimizes the first of these functions is called the minimax strategy, for which  $n = \tau$ . The quantity  $e = 1 - (n + \tau)$  is frequently used to characterize the efficiency of a prediction; it is the higher the closer  $e$  is to 1. An example of  $\varphi$  expressed in terms of damage is

$$\varphi = \alpha\lambda n + \beta\tau, \quad (4)$$

where  $\lambda$  is the rate of target events,  $\alpha$  is the cost resulting from a failure-to-predict,  $\beta\Delta$  is the cost of maintaining an alarm during  $(t, t + \Delta)$ . Therefore, (4) gives the loss rate entailed by  $\pi$ .

We now describe the structure of the optimal strategy. Let

$$r(t) = \lim_{\Delta \rightarrow 0} P\{\Delta N(t) > 0 \mid I(t)\}/\Delta$$

be the conditional rate of target events given  $I(t)$ . The optimal strategy in the problem (3) then declares an alarm every time  $r(t)$  exceeds a threshold  $r_0$ . The threshold is  $r_0 = \beta/\alpha$  when (4) is used. In the general case of  $\varphi(n, \tau)$ , we have to find the level  $c$  such that the line  $\{\varphi = c\}$  is tangent to the error diagram  $\Gamma$  (see Fig. 1). Suppose this occurs at a point  $Q = (n_0, \tau_0)$ . Then

$$r_0 = -\lambda \frac{dn}{d\tau}(Q) = -\lambda \frac{\partial \varphi}{\partial \tau} \bigg/ \frac{\partial \varphi}{\partial n}(Q)$$

where  $dn/d\tau$  is the slope of  $\Gamma$  at  $Q$ .

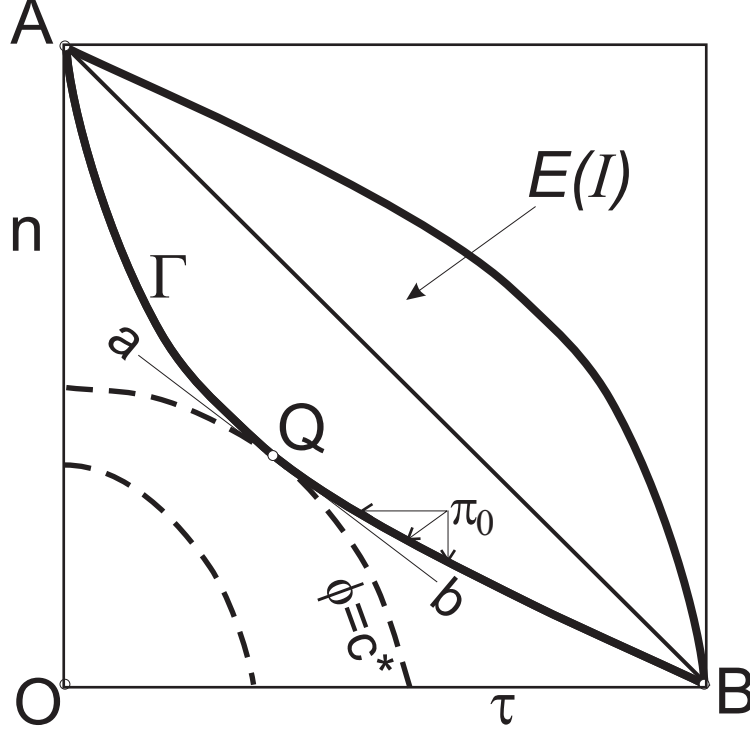


Figure 1: Error set  $\mathcal{E}(I)$  for prediction strategies based on a fixed type of information  $I = \{I(t)\}$ . The point  $A$  corresponds to an optimistic strategy, the point  $B$  to a pessimistic strategy, the diagonal  $D = AB$  corresponds to strategies of random guess.  $\Gamma$  is the error diagram of optimal strategies. *Small arrows* indicate strategies better than  $\pi_0$ , i.e. strategies with  $n \leq n(\pi_0)$  and  $\tau \leq \tau(\pi_0)$ . *Dashed lines* are isolines of a loss function  $\varphi(n, \tau)$ ; the isline of level  $c^*$  is tangent to  $\Gamma$  at the point  $Q$ , which corresponds to the optimal errors in the problem (3). The line  $(a, b)$  is tangent to  $\Gamma$  at  $Q$  and separates the two convex sets  $\mathcal{E}(I)$  and  $\{\varphi \leq c^*\}$ .

*The Relation to Hypothesis Testing.* We remind a classical hypothesis testing problem in mathematical statistics (see, e.g., Lehmann, 1959). Consider an observation  $\xi$ , which may be a scalar, a vector, or a functional object.

It belongs to the population with distribution  $P_0(dx)$  (hypothesis  $H_0$ ) or to the population with distribution  $P_1(dx)$  (hypothesis  $H_1$ ). A decision  $\pi(\xi) = 0$  or 1 in favor of  $H_0$  or  $H_1$ , respectively, entails errors of two kinds, viz.,

$$\alpha = P_0\{\pi(\xi) = 1\} \quad \text{and} \quad \beta = P_1\{\pi(\xi) = 0\}.$$

Let us fix  $\alpha$  and minimize the error  $\beta$  by a suitable choice of  $\pi$ . The lemma of J. Neyman and E. Pearson reads that, under certain regularity requirements, the optimal rule is such that  $\pi(\xi) = 1$ , as soon as

$$\mathcal{L}(\xi) = P_1(dx)/P_0(dx)|_{x=\xi} \geq c(\alpha),$$

otherwise  $\pi(\xi) = 0$ ; note that the threshold depends on  $\alpha$ .

In applications the power of the optimal test,  $1 - \beta$ , is considered as a function of  $\alpha$ , and called the Relative Operating Characteristic (ROC), see [Swets, 1973].

The prediction problem (3) is remarkable in that it can be interpreted in terms of hypothesis testing, so that the characteristics  $(n, \tau)$  become errors of the two kinds [Molchan, 2002]. The crucial observation for this is the following: the globally (in time) optimal strategy in (3) consists of locally optimal decisions on small time intervals  $(t, t + \Delta)$ . One can therefore disregard the global prediction problem and consider it on the interval  $(t, t + \Delta)$ . In this case  $\pi(t)$  interprets incoming information  $\xi = I(t)$  in terms of whether a target event will or will not occur in the interval  $(t, t + \Delta)$ . The characteristics  $(n, \tau)$  become errors of the two kinds, if  $P_0$  is the natural probability measure for the data  $I(t)$  at time  $t$ , while  $P_1(dx)$  is the conditional measure for  $I(t)$  given  $dN(t) = 1$ .

Recalling the definition of the risk function  $r(t)$ , one has

$$\begin{aligned} P_1(dx) &= P\{dN(t) = 1, I(t) \in dx\} / P\{dN(t) = 1\} \\ &= P\{dN(t) = 1 \mid I(t) = x\} P_0\{I(t) \in dx\} / P\{dN(t) = 1\} \\ &= r(t) P_0(dx) / \lambda. \end{aligned}$$

Hence  $P_1(dx)/P_0(dx) = r(t)/\lambda$ . Furthermore, since  $n$  and  $\tau$  are identical with the errors arising from testing  $H_1$  vs.  $H_0$ , we have

$$\text{ROC} = \{(1 - n, \tau) : (n, \tau) \in \Gamma\} = \Gamma^c$$

that is, the curves ROC and  $\Gamma$  are dual.

For this reason  $\Gamma^c$  is sometimes called ROC and sometimes the Molchan diagram. However, these names have different implications. The first name (ROC) always focuses our attention on errors of two kinds in a statistical problem, while the names  $n$  &  $\tau$  or error or Molchan diagram emphasize the connection between two of the many characteristics of prediction. The ROC interpretation of the curve  $\Gamma$  is possible thanks to specific features of the goal function and to the structure of the globally optimal strategy. With a modified goal function, the error diagram loses its relation to optimal strategies. The reason for this is that locally optimal decisions do not generally constitute the globally optimal strategy [Molchan & Kagan, 1992].

In this context we mention the case of prediction for an inhomogeneous Poisson process with a periodic rate function. It is commonly thought that the prediction of a Poisson process is trivial, and therefore does not deserve consideration. Molchan [1997, 2002] showed that this is not true, if the losses also include some cost for each switching from alarm to nonalarm and back again. An optimization problem of this kind is reasonable to avoid the cry wolf attitude.

Leaving aside the unimportant discussion of a suitable name for the  $n$  &  $\tau$  diagram, we put new questions: what are the analogues of  $\Gamma$  and  $D$  for space-time prediction? What is the structure of the optimal strategy for a goal function that is similar to (3)?

### 3 Space-time Prediction

For a theoretical analysis of prediction of large events in space-time it is sufficient to divide region  $G$  into disjoint parts  $G_i$  and to consider the vector point process

$$dN(t) = \{dN^{(1)}(t), \dots, dN^{(k)}(t)\},$$

where the component  $dN^{(i)}(t)$ ,  $P(\Delta N^{(i)}(t) \geq 2) = o(\Delta)$  describes the time sequence of target events in subregion  $G_i$ . In that case a prediction strategy  $\pi(t) = \{\pi^1(t), \dots, \pi^k(t)\}$  consists of the sequence of decisions

$$\pi^{(i)}(t) = \begin{cases} 1 & \text{alarm in } G_i \times \Delta t \\ 0 & \text{no alarm in } G_i \times \Delta t, \end{cases}$$

the decisions being based on the information  $I(t)$ .

Again the prediction results will be characterized by (1) and (2). Here,  $\tau_T = (\tau_T^1, \dots, \tau_T^k)$  is a vector whose  $i$ -th component defines the ratio of alarm time in  $G_i$  during time  $T$ . When the vector process  $(dN(t), I(t), \pi(t))$  is ergodic and stationary, the numbers  $n_T$  and  $\tau_T$  have the deterministic limits  $n \in [0, 1]$  and  $\tau \in [0, 1]^k$ , respectively, as  $T \rightarrow \infty$ .

The use of all possible strategies  $\pi$  based on  $I = \{I(t)\}$  yields the error set  $\{(n, \tau)\} = \mathcal{E}$  as a subset of the cube  $[0, 1]^{k+1}$ .

The set  $\mathcal{E}$  is convex. This can be demonstrated as follows. Having two strategies,  $\pi_1$  and  $\pi_2$ , with the characteristics  $(n, \tau)_i, i = 1, 2$ , we can devise a new one with the errors  $(n_1p + n_2q, \tau_1p + \tau_2q)$ , where  $p + q = 1$  and  $0 \leq p \leq 1$ . To do this, it is sufficient at every time step to use  $\pi_1(t)$  and  $\pi_2(t)$  in a random manner, with probabilities  $p$  and  $q$ , respectively. Changing  $p$  from 0 to 1, we get the straight segment that belongs to  $\mathcal{E}$  and connects  $(n, \tau)_1$  with  $(n, \tau)_2$ . Therefore,  $\mathcal{E}$  is convex.

The set  $\mathcal{E}$  always contains the following simplex:

$$\tilde{D}: \quad n + \langle \lambda, \tau \rangle / \Lambda = 1, \quad (n, \tau) \in [0, 1]^{k+1}, \quad \Lambda = \sum \lambda_i, \quad (5)$$

where the  $\lambda = (\lambda_1, \dots, \lambda_k)$  are the rates of target events in the subregions  $\{G_i\}$ , and  $\langle a, b \rangle = \sum a_i b_i$ . Equation (5) is satisfied by the following strategy based on trivial information. Let us declare an alarm during  $(t, t + \Delta)$  in subregion  $G_i$  with probability  $p_i$ ,  $\sum p_i \leq 1$ . Then the success rate in  $G_i$  is  $\lambda_i p_i / \Lambda$ . Therefore, we have  $n = 1 - \langle \lambda, p \rangle / \Lambda$  and  $\tau_i = p_i$ , i.e., (5) becomes an identity.

The simplex (5) is an analogue of the diagonal  $D$  used in the time prediction. The boundary of the convex set  $\mathcal{E}$  that lies below the plane (5) will be denoted  $\tilde{\Gamma}$  and termed the error diagram as above. We shall show that the diagram defines optimal strategies. To do this, we consider a function  $\varphi(n, \tau) > 0, \tau = (\tau^1, \dots, \tau^k)$  that is increasing with respect to each argument, and require that the level sets  $\{\varphi(n, \tau) \leq c\}$  be convex for any  $c > 0$ . Now we define the goal of time-space prediction using (3) with  $\tau = (\tau^1, \dots, \tau^k)$ .



Denote  $r^{(i)}(t) = \lim_{\Delta \rightarrow 0} P\{\Delta N^{(i)}(t) = 1 \mid I(t)\}/\Delta$ , the conditional rate of target events in subregion  $G_i$ , given the information  $I(t)$ , and denote by  $\lambda_i$  the unconditional rate.

**Statement 1.** *The optimal strategy for the space-time prediction with the goal function (3) declares an alarm in  $G_i \times [t, t + \Delta]$  as soon as*

$$r^{(i)}(t) > r_0^{(i)}$$

*and does not declare otherwise.*

*The thresholds are  $r_0^{(i)} = \beta_i/\alpha$ , if*

$$\varphi(n, \tau) = \alpha \Lambda n + \langle \beta, \tau \rangle. \quad (6)$$

*For the general case of  $\varphi(n, \tau)$ , we consider the level  $c$  such that the surface  $\varphi(n, \tau) = c$  is tangent to  $\tilde{\Gamma}$  at a point  $Q$ . Then*

$$r_0^{(i)} = -\Lambda \frac{\partial n}{\partial \tau^i}(Q) = -\Lambda \frac{\partial \varphi}{\partial \tau^i} \bigg/ \frac{\partial \varphi}{\partial n}(Q), \quad (7)$$

*where  $n = n(\tau^1, \dots, \tau^k)$  is the  $\tilde{\Gamma}$  function. Conversely, for any point  $Q = (n, \tau) \in \tilde{\Gamma}$  we can find the loss function  $\varphi(n, \tau)$  for which  $Q$  is optimal, i.e.,*

$$\varphi(Q) = \inf_{\{\pi\}} \varphi(n(\pi), \tau(\pi)),$$

*where the strategies  $\pi$  are based on  $I = \{I(t)\}$ .*

**Remark.** All components of the optimal strategy are interconnected due to the data  $I(t)$  which are common to subregions  $\{G_i\}$ .

**Proof.** Since  $(dN(t), I(t), \pi(t))$  is ergodic, the time average (1) can be identified with the ensemble average (over  $I(t)$ ) of the single term in (1) related to the interval  $(t, t + \Delta)$ . The same holds for (2) because  $N(T)/S \rightarrow \Lambda$  as  $T \rightarrow \infty$ . From this it follows that the globally optimal strategy for (3) can be derived by optimizing the decision in every interval  $(t, t + \Delta)$ . Putting  $\mathbf{1} = (1, \dots, 1)$ , we have

$$\begin{aligned}
n &= \lim_{\Delta \rightarrow 0} E < \mathbf{1} - \pi(t), \Delta N(t)/\Delta > / \Lambda = \\
&= \lim_{\Delta \rightarrow 0} E \{ E < \mathbf{1} - \pi(t), \Delta N(t)/\Delta > | I(t) \} / \Lambda = \\
&= E < \mathbf{1} - \pi(t), r(t) > / \Lambda = 1 - E < \pi(t), r(t) > / \Lambda, \\
\tau &= E\pi(t).
\end{aligned}$$

Here,  $\tau, \pi \in [0, 1]^k$ . Suppose  $\varphi(n, \tau)$  is of the linear form (6). Then

$$\varphi(n, \tau) = \alpha\Lambda + E < \pi(t), \beta - \alpha r(t) > . \quad (8)$$

The components of  $\pi(t)$  take on values in  $[0, 1]$ . Obviously, (8) has the least value, when we put

$$\pi^{(i)}(t) = \begin{cases} 0, & \beta^i - \alpha r^i(t) \geq 0 \\ 1, & \beta^i - \alpha r^i(t) < 0. \end{cases} \quad (9)$$

Suppose now that  $\varphi(n, \tau)$  is a nonlinear increasing function with convex level sets,  $\{\varphi \leq a\}$ . Then there exists a level  $c$  such that the surface  $\varphi(n, \tau) = c$  is tangent to  $\tilde{\Gamma}$  at some point  $Q$ . By the definition of  $\tilde{\Gamma}$ ,  $c$  is the least value of the goal function given the predictive information  $\{I(t)\}$ . Let us construct a plane that is tangent to  $\tilde{\Gamma}$  at the point  $Q$ ,  $an + < b, \tau > = c$ . It separates  $\tilde{\Gamma}$  and the surface  $\varphi(n, \tau) = c$ , because  $\mathcal{E}$  and  $\{\varphi(n, \tau) \leq c\}$  are convex. Therefore, the minimization of  $\varphi$  is equivalent to the minimization of the linear function  $an + (b, \tau)$ . The use of (8) and (9) yields (7). Actually, we have also proved the final part of the statement, because at any point  $Q \in \tilde{\Gamma}$  there exists a plane of support to  $\tilde{\Gamma}$ .  $\diamond$

**Prediction efficiency.** At the research stage of prediction, the efficiency of a time-space strategy  $\pi$  is sometimes characterized by the quantity  $e = 1 - (n + \tilde{\tau})$ , where

$$\tilde{\tau} = \sum_{i=1}^k \lambda_i \tau^{(i)} / \Lambda \quad (10)$$

is the rate of space-time alarm measured in terms of the rate of target events,  $\{\lambda_i\}$ . One can suggest some reasons in favor of this choice of  $e$ .

First,  $|e| \leq 1$  where  $e = 0$  for all trivial strategies, i.e.,  $(n_\pi, \tau_\pi) \in D$ , and  $e = 1$  for the ideal strategy with zero errors.

Second,  $e = (1-n) - \tilde{\tau}$ . In this identity the second term  $\tilde{\tau} = \tilde{\tau}(\pi)$  coincides with the rate of target events, which can be predicted by chance using the same space-time alarm characteristics  $(\tau^{(1)}, \dots, \tau^{(k)})$  as  $\pi$  has. Therefore,  $e$  determines the rate of nonrandom successes of the strategy  $\pi$ .

Third,  $e = e(\pi)$  is proportional to the Euclidian distance,  $\rho(Q, \tilde{D})$ , between  $Q = (n, \tau)$  and  $\tilde{D}$ ; moreover,  $e = 1$  for the ideal strategy having  $(n, \tau) = (0, 0) = O$ . Therefore,  $e(\pi) = \rho(Q, \tilde{D})/\rho(O, \tilde{D})$ , i.e.,  $e$  is the relative distance between  $\pi$  and the trivial strategies set in the coordinates  $(n, \tau_1, \dots, \tau_k)$ .

Our interpretation of  $e$  does not depend on the space parameter  $k$ . This is important for the comparison of predictions, because the space partition  $\{G_i\}$  is an independent element of a prediction strategy.

Fourth,  $e$  has the following additivity property:

$$e = 1 - n - \tilde{\tau} = \sum_{i=1}^k (1 - n_i - \tau_i) \lambda_i / \Lambda = \sum_{i=1}^k e_i \lambda_i / \Lambda,$$

where  $(n_i, \tau_i)$  and  $e_i = 1 - n_i - \tau_i$  are respectively the errors and the efficiency of  $\pi$  in subregion  $G_i$ . This follows from (10) and the relation

$$n = \sum_{i=1}^k n_i \lambda_i / \Lambda.$$

Thus,  $e(\pi)$  is a weighted mean of the efficiencies in subregions  $\{G_i\}$ . The additivity of  $e$  holds only for linear functions of the type  $e = an + b\tilde{\tau} + c$  (see Appendix 1 for exact formulation and proof).

To optimize  $e = 1 - (n + \tilde{\tau})$ , we must, in accordance with Statement 1, declare an alarm in  $G_i \times \Delta t$ , as soon as the probability gain  $(PG)$ ,  $r_\xi^{(i)}/\lambda_i$ , exceeds the level 1. This level is a point of equilibrium of  $PG$ , therefore, the alarm which optimizes  $e$  can be unstable in the general case of  $\{I(t)\}$ .

The following example is relevant to the stable situation [Molchan, 2002].

*Example 1 (characteristic earthquakes).* Consider the time prediction problem in which  $I(t)$  is the time  $u = t - t_k \geq 0$  that has elapsed since the last event  $t_k$ . In that case the optimal strategy for  $e = 1 - n - \tau$  declares an alarm in the interval  $(t_k, t_{k+1})$  as soon as

$$mF'(u)/(1 - F(u)) > 1, \quad t = t_k + u,$$

where  $F$  is the distribution of  $\Delta_k = (t_{k+1} - t_k)$  and  $m = E\Delta_k$  [Molchan, 2002]. In many interesting cases  $F'/(1 - F)$  has at most one extremum in the open interval  $(0, \infty)$ . Therefore the optimal alarm in  $(t_k, t_{k+1})$  consists at most of two intervals. It is easy to see that

$$e = \int_0^\infty [F'(u) - (1 - F(u))/m]_+ du,$$

where  $[a]_+ = a$ , if  $a > 0$  and  $[a]_+ = 0$ , if  $a < 0$ . The following table presents values of  $e$  depending on the coefficient of variation  $V = \sigma/m$  ( $\sigma^2$  is the variance of  $F$ ) for three types of distributions  $F$ , viz., Weibull ( $F(x) = 1 - \exp(-\lambda x^\alpha)$ ), Log-Normal, and Gamma ( $F'(x) = cx^{\alpha-1} \exp(-\lambda x)$ ):

$V$	.25	.50	.75
$e$	.52 - .60	.32 - .38	.15 - .22

Here all distributions have the same  $m$  and  $V$  parameters. Note that  $V \simeq 0.6$  for segments of the San Andreas fault, and that the model has a direct relation to the prediction of characteristic earthquakes. Therefore, our example with nontrivial prediction can be of interest for comparison with other available prediction methods.

**Trivial Strategies.** In the time prediction case the trivial strategies are described by the diagonal  $n + \tau = 1$  of the square  $[0, 1]^2$ . The end points  $(1, 0)$  and  $(0, 1)$  correspond to the so-called *optimistic* and *pessimistic* strategies (see Fig. 1). A pessimist maintains alarm all the time, while an optimist never uses it. These strategies are remarkable, because in a *regular situation* the points  $(1, 0)$  and  $(0, 1)$  are also the end points of the curve  $\Gamma$ , that is, trivial strategies may well be optimal ones. To understand the regular situation better, we consider the following counterexample.

*Example 2 (nonregular  $\Gamma$ ).* Let us consider the following model of target events:

$$dN(t)/dt = \sum_k \delta(t - t_k) + \sum_k \varepsilon_k \delta(t - t'_k), \quad t'_k = t_k + 1. \quad (11)$$

Here  $t_{k+1} - t_k \geq a > 0$  are i.i.d. random variables with the mean  $E(t_{k+1} - t_k) = m$  and  $\{\varepsilon_k\}$  are independent binary random variables with the distribution  $P(\varepsilon_k = 1) = p$ ,  $P(\varepsilon_k = 0) = 1 - p$ . In this model there are two types of target events, viz., main shocks  $\{t_k\}$  and reshocks  $\{t'_k = t_k + 1\}$  that may or may not occur.

To predict  $t_{k+1}$  using  $I(t) = \{t_p : t_p < t\}$  and  $t_k < t < t_{k+1}$  it is sufficient to declare an alarm at the moment  $t_k + a$  and cancel it after  $t = t_{k+1}$ . The reshock  $t'_k$  is predicted by short-term alarm at the moment  $t_k + 1 - 0$ . Now it is not difficult to see that the end points  $(n, \tau)$  of  $\Gamma$  are  $((1 + p)^{-1}, 0)$  and  $(0, 1 - a/m)$ . These points correspond to the regular situation, provided that  $p = a = 0$ .  $\diamond$

In the case of space-time prediction, the trivial strategies are described by the equation  $n + \tilde{\tau} = 1$ ,  $0 \leq n, \tau_i \leq 1$ . All solutions to that equation are obtained as the convex hull of extreme points  $(n, \tau_i = \varepsilon_i, i = 1, \dots, k)$ , where  $\varepsilon_i = 0$  or  $1$ , and  $n = 1 - \tilde{\tau}$ .

By definition we are in *the regular situation*, if all extreme points of  $\tilde{D}$  belong to  $\tilde{\Gamma}$ . This is true, if and only if  $I = \{I(t)\}$  is regular in each sub-region  $G_i$ ,  $i = 1, \dots, k$ . In the regular situation, strategies that maintain a continual alarm in part of the area of interest and no alarm in its supplement are optimal and trivial at once. This type of strategies includes Kullback's strategy [Kullback, 1959] ("relative intensity" in the terminology of Holliday et al. [2005]). The principle of the strategy is as follows. Suppose we know the epicenter density of target events,  $f(g)$ . Find the locations where  $f > c$  and declare a continual alarm there. This strategy is used during the research stage in order to minimize the alarm space volume.

In the polemical paper by Marzocchi et al. (2003), the Kullback strategy is used for comparison with the  $M8$  algorithm in the prediction of  $M \simeq 8$  ( $M \simeq 7.5$ ) earthquakes worldwide. Note that the Kullback strategy has  $n + \tilde{\tau} = 1$ . Therefore, the relative predictive potential of the  $M8$  algorithm can be measured by the quantity  $e = 1 - (n + \tilde{\tau})$ . To estimate  $\tilde{\tau}$  in a robust manner, we come to a nontrivial problem: to what degree can low magnitude seismicity (say,  $M = 4; 6$ ) be helpful in estimating the distribution  $f(g)$  (see  $\lambda_i/\Lambda$  in (10))? The problem is simpler for the case of predicting  $M = 7.5; 8$  along the Pacific Belt, because one has to compare smoothed one-dimensional seismicity distributions along the belt. This important problem unfortunately remains unexplored.

For the moment one can obtain only a rough estimate for the variability of  $n + \tilde{\tau}$  in the  $M8$  case. Denoting by  $N(T)$  the number of target events

for the monitoring period  $T$ , we find that a failure-to-predict will alter  $n$  by the amount  $\delta n \approx 1/N$  (10% in the prediction of  $M = 8$ ). According to [Kossobokov, 2005],  $\tilde{\tau}$  in the prediction of  $M = 7.5; 8$  varies within 5 – 10% when  $M = 4, 5, 6, 7$  is used to estimate the density of target events. Consequently, the variability of  $n + \tilde{\tau}$  for  $M = 8$  does not exceed 20%.

## 4 Prediction versus Forecasting

According to Statement 1, the prediction problem considered in its simplest version can be split into two. The one consists in estimating the conditional rate  $r(t, g, M)$  of magnitude  $M$  events in a space-time bin  $dg \times dt$ , while the other reduces to choosing a threshold  $r_0(g)$  for  $r(t, g, M)$ . This is an important conclusion for prediction practice, since the first problem is in the seismologist’s full competence, while the second is at the option of the customer. At first sight, the seismologist has merely to focus his efforts on the problem of estimating the risk function  $r(t, g, M)$ , i.e., on the *forecasting* problem.

In our view forecasting is different from prediction in that it involves no decisions, and prediction statements are probabilistic in character, namely, a target event  $M$  is expected to occur in the bin  $dg \times dt$  with some probability  $P(dg, dt)$ . For the small-bin case,  $P(dg, dt) \simeq r(t, g, M)dt dg$ .

At the present time, forecasting dominates the problem of earthquake prediction. Prediction proper came to be viewed as a binary forecasting, where there is no problem of choosing the thresholds. This transformation of the original prediction problem calls for some discussion.

When the information  $I(t)$  consists of an earthquake catalog, the problem of modeling  $r(t, g, M)$  is equivalent to constructing a model of the seismic process in the phase space  $(t, g, M)$  in terms of conditional rate. An example is the self-exciting model (ETAS as it is called today).

Substitution of forecasting for prediction raises a key question: what model of  $r(t, g, M)$  inspires greater confidence? In prediction, the information  $I(t)$  is chosen and transformed in such a way as to detect characteristic patterns premonitory to individual target events. At the research stage, the prediction is thought to be the better, the smaller the errors  $n$ ,  $\tilde{\tau}$ , or the combination  $n + \tilde{\tau}$ , say.

In forecasting, the goal is hazy; forecasting based on the conditional rate  $r(t, g, M)$  is considered to be the better, the better is an agreement between

the model of  $r$  and seismicity observed during a test period. Target events are rare as a rule, while premonitory phenomena are weak. For that reason the contribution of the latter into the fitting of the model of  $r$  is small too. Therefore a "good model" of seismicity will be determined mainly by typical seismicity patterns, such as clustering and aftershocks, regardless of whether they are premonitory or not. Under these conditions it is difficult to expect that the "good model" can automatically possess predictive properties in relation to large earthquakes. Therefore, having formally set up thresholds for  $r(t, g)$ , we shall arrive at errors  $n, \tilde{\tau}$  that are close to the diagonal  $n + \tilde{\tau} = 1$ , i.e., will obtain a misleading "objective proof" that large events are unpredictable.

The ETAS model is often considered to be the most suitable for description of seismicity [Ogata, 1999; Kagan and Jackson, 2000]. It is defined in a form convenient for prediction, in terms of the risk function

$$r(t, g, M) = \sum_{t-T < t_i < t} U(t, g, M | t_i, g_i, M_i) + U_0(g, M) \quad (12)$$

Here,  $U \geq 0$  is the conditional rate of first-generation aftershocks for an event  $(t_i, g_i, M_i)$ , and  $U_0 \geq 0$  is the rate of main shocks. The parameterization of  $U$  and  $U_0$  used in (12) is too simplistic for prediction purposes.

The ETAS model satisfactorily incorporates the clustering of events, hence it is convenient for describing aftershocks. It is known that some target events were preceded by patterns like seismicity increase and quiescence. When a threshold  $r > r_0$  is defined, the model (12) will respond to seismicity increase, but not to quiescence. The values of  $r$  are small in quiescent areas. Ogata [1988] tried to adapt (12) to deal with prediction of large events. In order to be able to respond both to seismicity increases and to quiescence, alarms were to be declared in two cases, when  $r$  was large and when  $r$  was small enough. This contradicts Statement 1. The use of two thresholds instead of a single one means that (12) is not the risk function for large events.

It thus appears that prediction of rare events need not rely on a detailed seismicity model. This can be seen from Example 1, when it is compared with results of the  $M8$  method, as well as from Statement 1, which asserts that detailed knowledge of  $r(t, g)/\lambda(g)$  is only needed about a fixed level  $c = 1$ . On the other hand, overfine detail in  $r/\lambda$  close to  $c = 1$  may inflate the number of false alarms. Considering forecasting instead of prediction,

we change the original goals and may misrepresent the predictability of rare events.

## 5 Predictability and Scale Invariance

Scaling laws are well known for seismicity: the distribution of events over energy (the Gutenberg-Richter law), the decay of seismicity in time following a large earthquake (the Omori law), the relation between source dimensions and earthquake energy, and spatial fractality of seismicity. The above list is being rapidly supplemented in recent years by laws that use scaling over different combinations of time, space, and energy. An example is the unified Bak law for the interevent time in a square of size  $L$  [Bak et al., 2002; Molchan and Kronrod, 2007]. Similarity ideas are actively used in the passage from the prediction of magnitude  $M$  to that of  $M - \Delta$ . The first attempt in this direction was for the CN algorithm (see, e.g., [Keilis-Borok and Rotwain, 1990]).

In the ideal case, if seismicity is strictly similar in the phase space  $(t, g, M)$ , the same predictability should be expected for  $M$  and  $M - \Delta$ . In particular, the events with  $M$  and  $M - \Delta$  are predictable or unpredictable at the same time based on the  $(t, g, M)$  data. The long-continued monitoring of target events using the M8 algorithm gives the following results [Kossobokov, 2005]: for the period 1985-2003 the error statistic  $n + \tilde{\tau}$  is equal to  $2/11 + 0.33 \simeq 0.5$  and  $22/52 + 0.34 \simeq 0.8$ , for  $M \simeq 8$  and  $M \simeq 7.5$ , respectively. The difference in  $n + \tilde{\tau}$  is substantial. If the difference is statistically significant, then it is natural to ascribe it to a violation of the similarity conditions. Indeed, the similarity condition for earthquakes is changed, when the source dimension is comparable with the width ( $W$ ) of the seismogenic lithosphere [Scholz, 1990; Pacheco et al., 1992; Okal and Romanowicz, 1994]. The  $M = 7.5, 8.0$  events fall in this category. Because  $W$  is subject to scatter worldwide, the finite-depth effect must be more relevant to  $M = 8$  events. There exist models for which one can neatly identify the size effect and its relation to predictability. Shapoval and Shnirman [2006] considered an avalanche model of the Bak type to show that events whose size is comparable with the size of the system are predictable similarly to the  $M = 8$  events in the M8 algorithm, i.e.,  $n + \tilde{\tau} \simeq 0.5$ . At the same time, the events that obey the power law distribution over energy are predicted much worse.

Whether the similarity conditions are violated is frequently inferred from



the presence of a bend in the Gutenberg-Richter frequency-magnitude relation. It is rather difficult to detect such a bend, especially in a regional environment. In that context we give a very simple example in order to demonstrate that the linearity of the frequency-magnitude relation does not preclude the predictability of individual magnitudes.

*Example 3 (predictability vs. GR law).* Consider a region where events with, say,  $M = 3, 4, 5$ , and 6 occur. The  $M = 3, 4$ , and 6 events are mutually independent in space-time. For the sake of simplicity we assume the distributions of all events to be uniform. Select 10% of the area,  $G$ , and require that each  $M = 5$  event in  $G$  be necessarily followed by a  $M = 6$  event during a time  $\delta$  (the location is left unspecified). This pattern allows the times of  $M = 6$  to be predicted based on the  $M = 5$  events. The prediction quality depends on the choice of  $\delta$ . At the same time, the frequency-magnitude law will hold in the entire area, if the rates for  $M = 3, 4$ , and 5 are  $\lambda(M) = a \cdot 10^{-M}$ . This relation is also true for  $M = 6$ , because one has

$$\lambda(M = 6) = \lambda(M = 5) \cdot 10^{-1} = a \cdot 10^{-6}.$$

by construction. The model has an obvious extension to the space-time prediction. Now since the  $M = 3, 4$  and  $M = 5$  events are independent, it follows that the  $M = 5$  events are unpredictable. The result is that, even though the Gutenberg-Richter law holds, only the  $M = 6$  events are predictable. This demonstrates that a violation of the similarity conditions need not entail changes in the Gutenberg-Richter law.

## 6 Conclusion

1. The simplest optimization problem of predicting the time of large events has been extended to the case of space-time prediction. We have found an analogue of the error diagram and described the optimal prediction strategies. The possibility and simplicity of this extension are due to a special choice of the class of goal functions (see (3)). In this particular case the globally optimal strategy can be constructed as a combination of locally optimal decisions. The situation becomes radically different, when the goal function is not a function of  $(n, \tau)$  alone.

2. The optimal prediction is split into two formally independent problems: modeling of the risk function  $r(t, g, M)$  and choosing its threshold. However,

a separate solution of these problems is a questionable way to real prediction.

3. In the theory presented here, the volume of space-time alarm  $A$  should be measured by the expected number of target events rather than geometrically as the product of area and time. Due to the simple statistical and geometric interpretation of  $e = 1 - n - \tilde{\tau}$ , this quantity is a natural candidate to represent the prediction efficiency at the research stage.

4. We demonstrate on an example that scaling laws in general do not exclude predictability of events of different magnitudes.

### **Acknowledgments**

This work was supported by the European Commission's Project 12975 (NEST) "Extreme Events: Causes and Consequences (E2-C2)" and in part by the Russian Foundation for Basic Research.

## REFERENCES

- Bak, P., Christensen, K., Danon, L., Scanlon, T., 2002. Unified Scaling Law for Earthquakes, *Phys. Rev. E*, **69**, 066106.
- Ellis, S.P., 1985. On optimal statistical decision rule for calling earthquake alerts. *Earthquake prediction Res.* **3**, 1-10.
- Holliday, J.R., Nanjo, K.Z., Tiampo, K.F., Rundle, J.B., Turcotte, D.L., 2005. Earthquake forecasting and its verification, *Nonlinear Processes in Geophysics*, *arXiv: cond-mat/0508476*.
- Kagan, Y.Y. & Jackson, D.D., 2000. Probabilistic Forecasting of Earthquakes, (Leon Knopoff's Festschrift), *Geophys. J. Int.* **143**, 438-453.
- Kossobokov, V.G., 2005. Earthquake Prediction: Principles, Implementation, Perspectives, in V.I. Keilis-Borok & A.A. Soloviev, Eds. *Computational Seismology*, Iss. **36-1**, 3-175, GEOS.
- Kossobokov, V. & Shebalin, P., 2002. Earthquake Prediction, in V.I. Keilis-Borok & A.A. Soloviev (eds), *Nonlinear Dynamics of the Lithosphere and Earthquake Prediction*, Springer, p.141-207.
- Keilis-Borok, V.I. & Rotwain, I.M., 1990. Diagnosis of times of increased probability of strong earthquakes in different regions of the world: algorithm CN, *Phys. Earth. Planet. Inter.*, **61**, 57-72.
- Keilis-Borok, V.I., Shebalin, P., Gabrielov, A., Turcotte, D., 2004. Reverse Tracing of Short-term Earthquake Precursors, *Phys. Earth. Planet. Inter.*, **145**, 75-85.
- Kullback, S., 1959. *Information Theory and Statistics*, J.Wiley & Sons.
- Lehmann, E.L., 1959. *Testing Statistical Hypotheses*, New York. J.Wiley & Sons.
- Lindgren, G., 1985. Optimal Prediction of Level Crossings in Gaussian Processes and Sequences. *Annals of Probability*, **13**:3, 804-824.
- Marzocchi, W., Sandri, L., Boschi, E., 2003. On the Validation of Earthquake-forecasting Models: the Case of Pattern Recognition Algorithms, *Bull. Seism. Soc. Am.*, **93**, 5, 1994-2004.

- Molchan, G.M., 1991. Structure of Optimal Strategies of Earthquake Prediction, *Tectonophysics*, **193**, 267-276.
- Molchan, G.M., 1997. Earthquake Prediction as a Decision Making Problem, *Pure Appl. Geophys.*, **149**, 233-247.
- Molchan, G.M., 2002. Earthquake Prediction Strategies: a Theoretical Analysis. In V.I. Keilis-Borok & A.A. Soloviev (eds), *Nonlinear dynamics of the Lithosphere and Earthquake Prediction*, Springer, p.209-237.
- Molchan, G.M. & Kagan, Y.Y., 1992. Earthquake Prediction and its Optimization, *J. Geophys. Res.*, **97**, 4823-4838.
- Molchan, G.M. & Kronrod, T.L., 2007. Seismic Interevent Time: A Spatial Scaling and Multifractality, *Pure Appl. Geophys.*, **164**, 75-96.
- Ogata, Y., 1988. Statistical Models for Earthquake Occurrences and Residual Analysis for Point Processes, *J. Amer. Stat. Association*, **83**, 401, 9-27.
- Ogata, Y., 1999. Seismicity Analysis Through Point-processes Modeling: A Review, *Pure Appl. Geophys.*, **155**, 471-457.
- Okal, E.A. & Romanowicz, B., 1994. On the Variation of  $b$ -values with Earthquake Size, *Phys. Earth. Planet. Inter.*, **87**, 55-76.
- Pacheco, J.F., Scholz, C.H., Sykes, L.R., 1992. Changes in Frequency-size Relationship from Small to Large Earthquakes, *Nature*, **355**, 71-73.
- Shapoval, A.B. & Shnirman, M.G., 2006. How Size of Target Avalanches Influences on Prediction Efficiency. *Int. J. of Modern Phys.*, **17**, 12, 1777-1790.
- Scholz, C.H., 1990. *The Mechanics of Earthquake and Faulting*, Cambridge Univ. Press, New York.
- Swets, J.A., 1973. The Relative Operating Characteristic in Psychology, *Science*, **182**, 4116, 990-1000.

## Appendix 1

The efficiency  $e = 1 - n - \tilde{\tau}$  belongs to the following class of continuous functions  $f(z)$ ,  $z = (x, y)$ : for any  $m$  and  $p = (p_1, \dots, p_m)$ ,  $\sum p_i = 1$ ,  $0 \leq p_i \leq 1$  there exist such  $\{a_i(p), i = 1, \dots, m\}$  that

$$f\left(\sum_1^m p_i z_i\right) = \sum f(z_i) a_i(p). \quad (A1)$$

Here  $z_i = (n_i, \tau_i)$  are errors relevant to the subregion  $G_i$ ,  $p_i = \lambda_i/\Lambda$ , and  $\sum_1^m p_i z_i = (n, \tilde{\tau})$ .

Let us prove that any continuous function  $f$  with the property (A1) is linear, i.e.,  $f(x, y) = ax + by + c$ .

It is enough to consider the case  $m = 2$ . One has

$$f(pz_1 + qz_2) = f(z_1)a(p) + f(z_2)b(p), \quad q = 1 - p. \quad (A2)$$

If  $f(z_0) \neq 0$ , then using limit  $z_i \rightarrow z_0$  one has

$$a(p) + b(p) = 1. \quad (A3)$$

Applying (A2) with  $p = q = 1/2$  to all  $z_1 : |z_1 - z_0| = R$ ,  $z_2 = 2z_0 - z_1$  and using (A3), we get

$$f(z_0) = \int_{|z-z_0|=R} f(z) ds.$$

Thus,  $f$  is a harmonic function; in particular,  $f$  is smooth.

Substitute  $z_1 = z_0 - kqz$ ,  $z_2 = z_0 - kpz$  in (A2) and differentiate (A2) with respect to  $k$  at  $k = 0$ . Then we get

$$0 = (-qa + pb) \cdot \rho,$$

where  $\rho = f'_n(z_0)x + f'_\tau(z_0)y$ ,  $z = (x, y)$ . If  $\rho \neq 0$ , we have  $b = q$  and  $a = p$ . By (A1),  $f$  is linear.